INTERNATIONAL JOURNAL FOR NUMERICAL AND ANALYTICAL METHODS IN GEOMECHANICS
Int. J. Numer. Anal. Meth. Geomech., **23,** 243–262 (1999)

A METHOD FOR THE ANALYSIS OF LARGE VERTICALLY LOADED PILE GROUPS

D. C. HONG, Y. K. CHOW* AND K. Y. YONG

Centre for Soft Ground Engineering, Department of Civil Engineering, National University of Singapore, Singapore 0511, Singapore

SUMMARY

An iterative method is described for the analysis of vertically loaded pile groups with a large number of vertical piles. The individual pile response is modelled using load-transfer (*t–z*) curves while pile–soil–pile interaction is determined using Mindlin's solution. The present method not only keeps all the advantages of the so-called 'hybrid method', but also makes it possible for practising engineers to solve problems of large non-uniformly arranged pile groups in a time-saving way using a personal computer. Good agreement between the present method of analysis and the direct method is observed. A case history is analysed and the computed response of a large pile group compares favourably with the field measurement. Copyright © 1999 John Wiley & Sons, Ltd.

Key words: deformation; pile group; computation; pile–soil-pile interaction

INTRODUCTION

Various theoretical methods have been reported to analyse vertically loaded pile groups. Theoretically, the finite element method can be used to obtain the response of pile groups embedded in any naturally occurring soils. However, a large amount of computer memory and computer central processor unit (CPU) time are required for the three-dimensional finite element analysis of pile groups, even for the assumption of linear soil behaviour (e.g. Reference 1). This makes the method practically inapplicable.

The boundary integral solutions involve only a discretization of the boundaries, hence the computer memory and CPU time in the analysis required are less. The earliest applications of this method were proposed by Butterfield and Banerjee[2] and Poulos[3]. In their methods, both the soil responses at individual piles and pile–soil–pile interaction are all determined using Mindlin's solution[4]. The method has subsequently been extended by Poulos[5] to include soil non-linearity and soil in-homogeneity in an approximate way.

A more rigorous integral equation method has been proposed by Banerjee *et al.*[6] to account for the influence of soil in-homogeneity and soil non-linearity. However, this method leads to to a rapid increase of the size of equations to be solved, and hence makes it less attractive.

* Correspondence to: Dr. Y. K. Chow, Department of Civil Engineering, National University of Singapore, Kent Ridge, Singapore 0511, Singapore

CCC 0363–9061/99/030243–20$17.50

Received 20 October 1997
Revised 10 April 1998

Chow[7] proposed a method for the analysis of axially and laterally loaded pile groups embedded in non-homogeneous soils. The group piles are discretized into discrete elements. The load–deformation relationship of the soil is determined using the flexibility approach, with the influence coefficients evaluated numerically using the finite element method with Fourier synthesis. The method provides a rigorous and economical treatment of the soil inhomogeneity.

O'Neill *et al.*[8] have proposed an iterative 'hybrid' method in which the individual pile response is modelled using load–transfer (t–z) curves and pile–soil–pile interaction is based on Mindlin's solution. The response of pile groups can be obtained through an iterative procedure in which the load–transfer (t–z) curves are adjusted to account for the group effect. A refinement of the 'hybrid' analysis was presented by Chow[9] whereby the true pile–soil–pile interaction is considered directly and the global stiffness equations for pile groups need to be solved. Following the analytical model by Randolph and Wroth[10] and the work of Kraft *et al.*[11], the non-linear behaviour of the soil was also considered.

Chow[12] has also proposed an iterative approach using the hybrid method. The individual pile stiffness matrices can be uncoupled and only the partition corresponding to a single pile in the global stiffness matrix is used for each iteration. Hence, the computer storage requirement is significantly reduced compared with the direct method. However, this iterative method may take more solution time than the direct method.

In this paper, an iterative method is described which basically is an improvement of the approach by Chow[12]. Only the stiffness equations of one individual pile in a pile group need to be solved at each iteration step. The quick convergence of the solutions is observed. Hence, the large amount of CPU time which otherwise is spent in solving the global stiffness equations of pile groups in direct methods is saved. The accuracy of this method is examined by comparison with the direct 'hybrid method' (i.e. Reference 9). Finally, the response of a large piled raft under vertical loading is analysed.

METHOD OF ANALYSIS

When a pile group is subject to vertical loads, the relationship between the settlements of pile heads and the vertical loads is the main concern of practising engineers. In this paper, a pile head flexibility matrix $[F_{\text{head}}]$ is introduced to represent such a relationship of interest. The coefficient F_{ij} in the matrix $[F_{\text{head}}]$ is the pile head flexibility coefficient which denotes the head settlement of pile i due to a unit vertical load acting on the head of pile j. Hence, the relationship between the pile head settlement and external vertical forces on the pile heads can be established as follows:

$$\begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1n} \\ F_{21} & F_{22} & \cdots & F_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{Bmatrix} = \begin{Bmatrix} w_1 \\ w_2 \\ \cdots \\ w_n \end{Bmatrix} \tag{1}$$

or

$$[F_{\text{head}}]\{p_{\text{head}}\} = \{w_{\text{head}}\} \tag{2}$$

where p_i is the vertical force on the head of pile i, w_i is the head settlement of pile i, $i = 1, 2, \ldots, n$, $\{p_{\text{head}}\}$ is the vector of pile head loading, $\{w_{\text{head}}\}$ is the vector of pile head settlements and n is the number of piles in the group.

The procedure to obtain the pile head flexibility coefficients in the matrix $[F_{\text{head}}]$ can be divided into n steps for a pile group with n piles. In each step, only a unit vertical load acts on only one pile head while the other piles in the group are free of external loading. Using the iterative procedure described below, the pile head settlements of the group piles due to a unit point load on any pile, say, pile m, can be determined (i.e. the coefficients of column m in the matrix $[F_{\text{head}}]$ can be obtained.). For the moment, the effect of a pile cap connecting the pile heads together is not considered. The extension to consider the effect of a rigid pile cap will be described later.

When a unit vertical load is applied on the head of pile m, this pile is treated as a single pile ignoring the existence of other piles to obtain the *initial* response of the pile. Using the conventional finite element method (e.g. Reference 13), the load–displacement relationship of a single pile, i.e. pile m, subject to vertical loading is given by

$$([K_{\text{p}}] + [K_{\text{s}}])\{w_{\text{ini.}}^{(m)}\} = \{P_{\text{e}}\} \tag{3}$$

where $[K_{\text{p}}]$ is the stiffness matrix of the elements of a single pile, $[K_{\text{s}}]$ the soil stiffness matrix (a practical formulation of soil stiffness matrix is to lump soil stiffness at the pile nodes which is adequate in more practice problems.), $\{w_{\text{ini.}}^{(m)}\}$ the vector of *initial* vertical deformations of pile m, and $\{P_{\text{e}}\}$ is the vector of external vertical loading.

As a result of the *initial* vertical deformations of pile m, the pile–soil interaction around pile m is mobilized which leads to soil displacements in places where other piles in the group are located. The soil displacement at any node of the other piles, say node i, due to the pile–soil interaction forces at the nodes of pile m can be obtained by superposition from

$$w_{0i}^{(1)} = \sum_{j=1}^{j=nn} f_{ij} P_{\text{s}j}^{(1)} \tag{4}$$

where $w_{0i}^{(1)}$ is the soil displacement at node i, f_{ij} is the flexibility coefficient denoting the soil displacement at node i due to a unit pile–soil interaction force at node j, $P_{\text{s}j}^{(1)}$ is the pile–soil interaction force at the node j of pile m, and nn is the number of nodes of pile m. The flexibility coefficients are obtained from Mindlin's solution for the influence of a unit point force in an isotropic, homogeneous elastic half-space. The extension to non-homogeneous soil can be implemented by means of an average procedure similar to that described by Poulos[5] and Chow[9]. The pile–soil interaction force at node j of pile m, $P_{\text{s}j}^{(1)}$, can be given by

$$P_{\text{s}j}^{(1)} = K_{hj}\, l w_{\text{ini.}\,j}^{(m)} \tag{5}$$

where $K_{hj}(= \pi d k_{hj})$ is the soil stiffness per unit length of the pile at node j, $\mathrm{w}_{\text{ini.}\,j}^{(m)}$ is the *initial* vertical deformation of pile m at node j, k_{hj} is the modulus of subgrade reaction of soil at node j, d is the diameter of pile and l is the pile element length associated with node j.

The induced soil displacements will then cause vertical deformations of the other piles in the group. Incorporating such an influence, the governing differential equation of a pile embedded in a soil modelled using load–transfer curves or modulus of subgrade reaction method is given by

$$-E_{\text{p}} A \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \pi \mathrm{d} k_h (w - w_0) = 0 \tag{6}$$

where w is the vertical deformation of pile, w_0 the soil displacement due to pile–soil interaction around other piles, E_{p} the Young's modulus of the pile material, A the cross-sectional area of pile, z the depth in soil and d the width or diameter of pile. For the moment, linear load–transfer curves

are constructed following the theoretical work of Randolph and Worth.[10] The non-linear soil behaviour can be incorporated using any non-linear load–transfer (t–z) curves. The determination of soil parameters for the construction of load–transfer (t–z) curves can follow the work of Chow.[9] The solution procedure to take into account non-linear soil behaviour has been described in Reference 9.

By applying Galerkin's method, Equation (6) yields the following element matrix equations:

$$([K_{\rm p}^{\rm e}] + [K_{\rm s}^{\rm e}])\{w_{\rm p}\} = \int_0^l K_h \{N\} w_0(z)\,{\rm d}z \tag{7}$$

where $[K_{\rm p}^{\rm e}]$ is the pile element stiffness matrix, $[K_{\rm s}^{\rm e}]$ the soil element stiffness matrix, $\{w_{\rm p}\}$ the vector of pile element vertical deformations, $K_h(=\pi d k_{\rm h})$ the soil stiffness per unit length of the pile, $\{N\}$ the vector of shape functions for the pile element subject to vertical loading, $w_0(z)$ the induced soil displacements due to the pile–soil interactions around other piles and l the length of the pile element. The vector on the right-hand side of Equation (7) represents the induced loading acting on the pile element resulting from soil displacements due to the pile–soil interaction forces around other piles.

By lumping the soil stiffness at pile nodes and assembling the elements matrices for all the pile elements, the global stiffness equations for the pile–soil system of a single pile subject to vertical soil displacements can be given by

$$([K_{\rm p}] + [K_{\rm s}])\{w_{\rm p}\} = [K_{\rm s}]\{w_0\} \tag{8}$$

where $\{w_{\rm p}\}$ is the vector of vertical deformations of nodes of a single pile, and $\{w_0\}$ is the vector of induced vertical soil displacement due to pile–soil interaction forces. It may be noted that $[K_{\rm s}]$ is the lumped soil stiffness matrix in this formulation.

For each individual pile, the *initial* response of all the piles except pile m can be obtained by Equation (8) with $\{w_0\}$ obtained from Equation (4). The *initial* response of the group piles due to a unit load on m, $\{w_{\rm ini.}\}$, are now determined.

It should be noted that this is not the *final* response of the piles since the pile–soil interactions around the influenced piles (i.e. all piles except pile m) can likewise cause soil displacements at the locations of the other piles which can lead to *additional* deformations of these piles. However, these pile–soil interaction effects do not produce as pronounced an influence as those caused by pile m. An interaction transfer parameter β, which should be less than 1·0 but larger than 0, is employed to account for such a reduced influence. The soil displacement at any node i, $w_{0i}^{(2)}$, due to the pile–soil interaction around all piles except for the soil reactions at pile m is given by

$$w_{0i}^{(2)} = \beta \sum_{j=1}^{j=nn} f_{ij} P_{{\rm s}j}^{(2)} \tag{9}$$

where β is interaction transfer parameter, f_{ij} is the flexibility coefficient denoting the soil displacement at node i due to a unit pile–soil interaction force at node j, $P_{{\rm s}j}^{(2)}$ is the pile–soil interaction force at node j, and nn is the total number of nodes in the group except those nodes of pile m and those nodes associated with the same pile as node i. The pile–soil interaction force at one node of any pile i, cannot induce additional soil displacements at the other nodes of pile i. This arises from the fundamental assumption of the subgrade reaction modulus method. The

flexibility coefficients are obtained from Mindlin's solution and pile–soil interaction force $P_{sj}^{(2)}$ is given by

$$P_{sj}^{(2)} = K_{hj} l (w_{\text{ini}.j} - w_{0j}^{(1)}) \tag{10}$$

where $w_{\text{ini}.j}$ is the *initial* deformation of pile at node j, $w_{0j}^{(1)}$ the induced vertical soil displacement at node j due to the pile–soil interaction caused by pile m, and l is the pile element length associated with node j. $w_{\text{ini}.j}$ is given by solving Equation (8) and $w_{0j}^{(1)}$ is given by Equation (4). An implication in Equation (10) is the compatibility of the displacements of piles and the soils around them.

The *additional* vertical deformations $\{w_{\text{add.}}\}^{(1)}$ of the piles in the group can then be obtained by solving Equation (8) for each individual pile while $\{w_0\}$ is now given by Equation (9). For the group piles, the vector of vertical deformations can be given from superposition

$$\{w\}^{(1)} = \{w_{\text{ini.}}\} + \{w_{\text{add.}}\}^{(1)} \tag{11}$$

where $\{w\}^{(1)}$ is the vector of vertical deformations of nodes of the group piles, the number in brackets on the upper right side, i.e. 1, is the iteration index, $\{w_{\text{ini.}}\}$ is the vector of *initial* vertical deformations of nodes of the group piles and $\{w_{\text{add.}}\}^{(1)}$ is the vector of *additional* vertical deformations of nodes of the group piles.

It should be noted that these *additional* vertical deformations $\{w_{\text{add.}}\}^{(1)}$ can lead to *additional* pile–soil interaction forces. And the *additional* pile–soil interaction forces can then induce *additional* soil displacements in the places of the other piles in the group which can likewise cause more vertical deformations of these piles. The *additional* soil displacement at any node i, $w_{0i}^{(3)}$, is given by

$$w_{0i}^{(3)} = \beta \sum_{j=1}^{j=nn} f_{ij} P_{sj}^{(3)} \tag{12}$$

Equation (12) is similar to Equation (9) except that nn is the total number of nodes in the group except those nodes associated with the same pile as node i and pile–soil interaction force $P_{sj}^{(3)}$ is given by

$$P_{sj}^{(3)} = K_{hj} (w_{\text{add}.j}^{(1)} - w_{0j}^{(2)}) \tag{13}$$

where $w_{\text{add}.j}^{(1)}$ is the *additional* pile deformation at node j and $w_{0j}^{(2)}$, which is given by Equation (9), is the induced vertical soil displacement at node j.

Another additional vertical deformations $\{w_{\text{add}.j}\}^{(1)}$ of the group piles can be determined by solving Equation (8) for each pile while the vector $\{w_0\}$ on the right-hand side is given by Equation (12). Equation (11) can now be re-written as

$$\{w\}^{(2)} = \{w_{\text{int.}}\} + \{w_{\text{add.}}\}^{(1)} + \{w_{\text{add.}}\}^{(2)} \tag{14}$$

where $\{w\}^{(2)}$ is the vector of vertical deformations of the group piles, $\{w_{\text{add.}}\}^{(1)}$ is the vector of the first *additional* vertical deformations of the group piles, and $\{w_{\text{add.}}\}^{(2)}$ is the vector of the second *additional* deformations of the group piles.

This pile–soil–pile interaction process will repeat until further interaction does not affect the response of each pile in the group, i.e. convergence of the solutions. To satisfy this requirement, a criterion is given as

$$\frac{w_i^{(k)} - w_i^{(k-1)}}{\mathrm{W}_i^{(k)}} < \varepsilon \tag{15}$$

where $w_i^{(k)}$, $w_i^{(k-1)}$ is the pile vertical deformation at node i ($i = 1, 2, \ldots, nn$), nn is the total number of node in the group, k is the iteration index and ε is the allowable tolerance.

It should be noted that for iterations $k \geqslant 2$, Equation (12) would be used to determine additional soil displacements due to pile–soil interaction forces.

When this criterion is satisfied, the *final* pile head settlements of the group under a unit vertical load on the head of pile m, i.e. $F_{1m}, F_{2m}, \ldots, F_{nm}$ (n is the number of piles in the group) in the pile head flexibility matrix $[F_{\text{head}}]$, are now determined. All the pile head flexibility coefficients in the matrix can likewise be obtained by putting a unit vertical load on the head of each pile in the group in turn.

In engineering practice, a rigid pile cap is used which imposes a uniform settlement of the pile heads. The pile head flexibility matrix $[F_{\text{head}}]$ can be inverted to give the relationship of the external loads and the pile head settlements as follows:

$$\{p_{\text{head}}\} = [F_{\text{head}}]^{-1}\{w_{\text{head}}\} \tag{16}$$

or

$$\{p_{\text{head}}\} = [K_{\text{head}}]\{w_{\text{head}}\} \tag{17}$$

where $[K_{\text{head}}] = [F_{\text{head}}]^{-1}$ is the pile head stiffness matrix.

Hence, a prescribed displacement can be implemented by using Equation (17). The subsequent analyses are based on the solutions for pile groups with a rigid pile cap which is assumed not to be in contact with the ground.

COMPARISON WITH THEORETICAL SOLUTIONS

The accuracy of the present method of analysis is assessed by comparison with the approach of Chow[9] which has been verified by comparison with the more rigorous integral equation method of Butterfield and Banerjee.[2] In this paper, the approach of Chow[9] will be called the 'direct method' because the global stiffness matrix of the pile–soil system of the pile group is assembled and then the vertical deformations of piles are obtained directly by solving the global equations. Ten discrete elements are used to model each individual pile in the analysis because ten elements would be sufficient to meet the requirement of accuracy for practical purposes.

The dimensionless parameters of interest here are p_{group}/Gr_0w, p/Gr_0w, λ, L/r_0, s/r_0 and v, where p_{group} is the total vertical load acting on the pile cap, p is the vertical load acting on the individual pile head, G is the soil shear modulus, r_0 is the pile radius, w is the settlement of pile head, $\lambda = E_p/G$ is the stiffness ratio, E_p is Young's modulus of the pile material, L/r_0 is the slenderness ratio, L is the penetration depth of the pile, s/r_0 is the normalized spacing of piles, s is the pile spacing (centre to centre) and v is the Poisson's ratio of soil.

In engineering practice, the arrangement of piles under a pile raft is not always uniform. Sometimes, piles are closely arranged under columns which transfer loads of superstructures to the pile foundation whereas a relatively loose layout of piles under the rest part of the pile raft is employed. Hence, the subsequent analyses do not take into account symmetry of pile groups. The extension of the present approach to take into account symmetry of pile groups will be given in Appendix I. The present approach can analyse the response of pile groups with piles of different length and diameter. However, the following analyses will focus on the response of pile groups with piles of same properties.

Table I. Normalized stiffness p_{group}/Gr_0w of 10×10 pile groups ($\lambda = 1000$, $s/r_0 = 5$, $\nu = 0{\cdot}5$ and $\beta = 0{\cdot}1$)

L/r_0	Allowable tolerance ε			Direct method
	0·005	0·01	0·05	
20	256	256	256	257
40	289	290	290	297
60	319	319	320	333
80	345	345	345	366
100	367	368	365	397

Table II. Solution time† (in min) for 10×10 pile groups using a personal computer‡ ($\lambda = 1000$, $s/r_0 = 5$, $\nu = 0{\cdot}5$ and $\beta = 0{\cdot}1$)

L/r_0	Allowable tolerance ε		
	0·005	0·01	0·05
20	3·5	3·5	3·5
40	5	5	3·5
60	7	5·5	3·5
80	8·5	8	5
100	12	10	5

† The solution times using the Direct method in the personal computer for all the cases are 44 min.
‡ Configuration of the personal computer: (a) 100 MHz Pentium-S, (b) 32 Mb core memory

Table I shows the group normalized stiffness of 10×10 square pile groups using different allowable tolerance ε in the present approach. The normalized stiffness of the groups given by the direct method are also shown for comparison. It can be seen that $\varepsilon = 0{\cdot}05$ is sufficient to give accurate solutions. The solution time using different allowable tolerance ε is given in Table II. Smaller ε values lead to more solution time while the accuracy of solutions does not improve. Hence, the following analysis is based on the convergence criterion of Equation (15) with an allowable tolerance $\varepsilon = 0{\cdot}05$.

As has been described earlier, n calculation steps are needed to determine the pile head flexibility coefficients of the matrix $[F_{head}]$ in Equation (2). A number of iterations are needed to achieve convergence of solutions for each calculation step. It should be noted that different numbers of iterations are required at different steps. Normally, more iterations are needed when a unit vertical load acts on a pile in the central region of a pile group. In other words, it will take less iterations to achieve convergence to determine the pile head flexibility coefficients related to the piles in the side region of the group. As a typical example, large pile groups with 250 piles are used. The layouts of the pile groups are roughly square. Tables III and IV show the number of iterations to achieve convergence for the solutions of the pile groups while using different interaction transfer parameters. It should be noted that different iterations are needed in some cases, e.g. the solutions for 224 of the piles needed three iterations and those for the remaining

Table III. Number of iterations for the solutions of pile groups with 250 piles† ($\lambda = 1000$, $s/r_0 = 5$ and $v = 0{\cdot}5$)

L/r_0	Interaction transfer parameter β				
	0·05	0·1	0·15	0·2	0·25
20	1 (250)	1 (250)	1 (250)	1 (250)	1 (250) 2 (21)
60	1 (250)	1 (250)	1 (71) 2 (179)	1 (4) 2 (246)	3 (57) 4 (113) 5 (59)
100	1 (86) 2 (164)	2 (26) 3 (224)	—‡	—‡	—‡

† The values in the parentheses are the numbers of the piles which are related to the number of iterations shown outside the parentheses
‡ Divergence of the solutions

Table IV. Number of iterations for the solutions of pile groups with 250 piles† ($\lambda = \infty$, $s/r_0 = 5$ and $v = 0{\cdot}5$)

L/r_0	Interaction transfer parameter β				
	0·05	0·1	0·15	0·2	0·25
20	1 (250)	1 (250)	1 (250)	1 (250)	1 (250)
60	1 (250)	1 (250)	1 (250)	1 (94) 2 (156)	—‡
100	1 (250)	1 (48) 2 (202)	—‡	—‡	—‡

† The values in the parentheses are the number of the piles which are related to the number of iterations shown outside the parentheses
‡ Divergence of the solutions

26 piles needed two iterations for the case of $L/r_0 = 100$ and $\beta = 0{\cdot}1$. The larger L/r_0 is, the more iterations are needed. Larger β leads to more iterations and hence more solution time is needed. In some cases, an inappropriate too large β may cause the divergence of solutions as shown in Tables III and IV in the cases of $L/r_0 = 100$ and $\beta \geqslant 0{\cdot}15$. The judgement of divergence of solutions is given as follows.

When a unit vertical load acts on the head of a pile and the other piles in a group are free of external loading, pile–soil interaction forces around all the piles in the group should be around 1. However, if divergence occurs, pile–soil interaction forces could be much more than 1, say, 1000, or much less than 1, say, -1000. Hence, an appropriate β should be used in the present approach.

The normalized stiffness of the groups for the problems shown in Tables III and IV are shown in Tables V and VI. It can be seen that β can be used over a wide range and the accuracy of solutions will not be affected very much by different β values as long as β is less than an upper

Table V. Normalized stiffness p_{group}/Gr_0w of pile groups with 250 piles ($\lambda = 1000$, $s/r_0 = 5$, and $v = 0{\cdot}5$)

L/r_0	Interaction transfer parameter β					Direct method
	0·05	0·1	0·15	0·2	0·25	
20	390	390	390	390	390	391
60	453	456	458	457	460	472
100	504	511	—†	—†	—†	545

† Divergence of the solutions

Table VI. Normalized stiffness p_{group}/Gr_0w of pile groups with 250 piles ($\lambda = \infty$, $s/r_0 = 5$, and $v = 0{\cdot}5$)

L/r_0	Interaction transfer parameter β					Direct method
	0·05	0·1	0·15	0·2	0·25	
20	395	395	395	396	396	395
60	475	478	480	481	481	489
100	557	556	—†	—†	—†	580

† Divergence of th solutions

limit beyond which divergence may occur. For engineering purposes, the actual upper limits of β need not be determined. Since small β values can save solution times, can give accurates solutions and furthermore, can give convergent solutions, a small interaction transfer parameter, i.e. $\beta = 0{\cdot}1$, will be used in the following analysis. As will be shown later, the use of $\beta = 0{\cdot}1$ can solve problems from 2 piles to 462 piles and good agreement between the present approach and the Direct method as well as a field measurement is observed.

Figures 1–5 show the load distribution and the load-settlement behaviour of individual piles within pile groups of different configurations. The agreement between the present approach and the direct method is good. In all cases, the results compared are for λ values of ∞ and 1000, with the soil Poisson's ratio $v = 0{\cdot}5$, and a normalized pile spacing $s/r_0 = 5$. In Figures 2–5, the layouts of 5×5, 10×10 and 15×15 pile groups are square. Pile 1 is the one which is at the corner of the groups, Pile 2 is the middle of the outer side row of the groups and Pile 3 is at the centre of the groups.

As has been described earlier only the stiffness equations of individual piles in pile groups need to be solved at each iteration step whereas the global stiffness matrix of pile group has to be assembled and solved in the Direct method. Hence, for a pile group with a large number of piles it will take much longer solution time in solving the global stiffness equations in the Direct method. Table VII shows a comparison of normalized stiffness of pile groups obtained by using the present approach and the Direct method. The plan arrangement of the pile groups is roughly square. The results compared are for $\lambda = 1000$, with the soil Poisson's ratio $v = 0{\cdot}5$, the slenderness ratio $L/r_0 = 60$, and a normalized pile spacing $s/r_0 = 5$. The difference between the two methods is less than 4 per cent for all the cases. Table VIII shows the solution times of the Direct

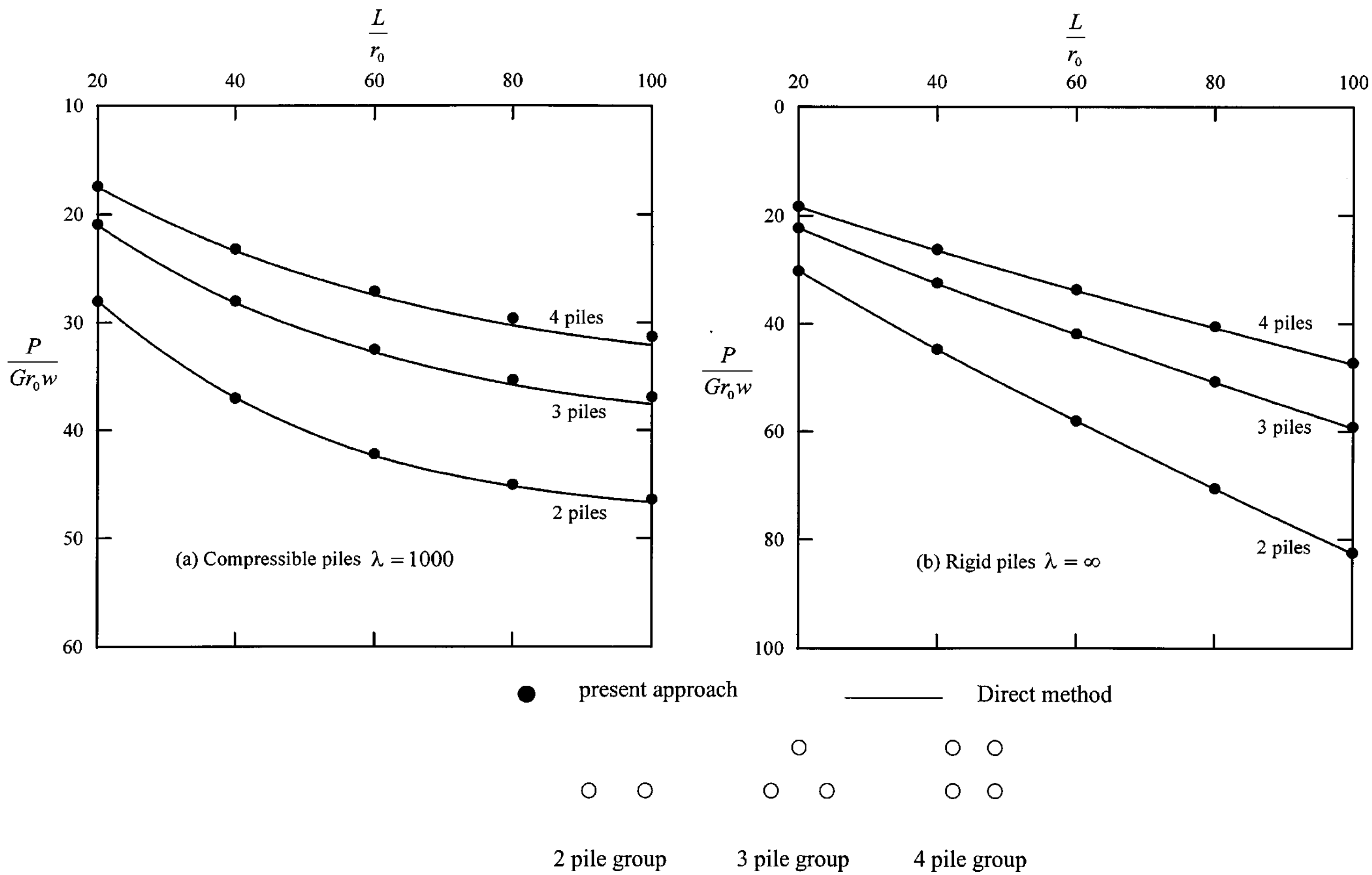

Figure 1. Comparison of load distribution for pile groups ($s/r_0 = 5$, $\nu = 0{\cdot}5$, $\beta = 0{\cdot}1$)

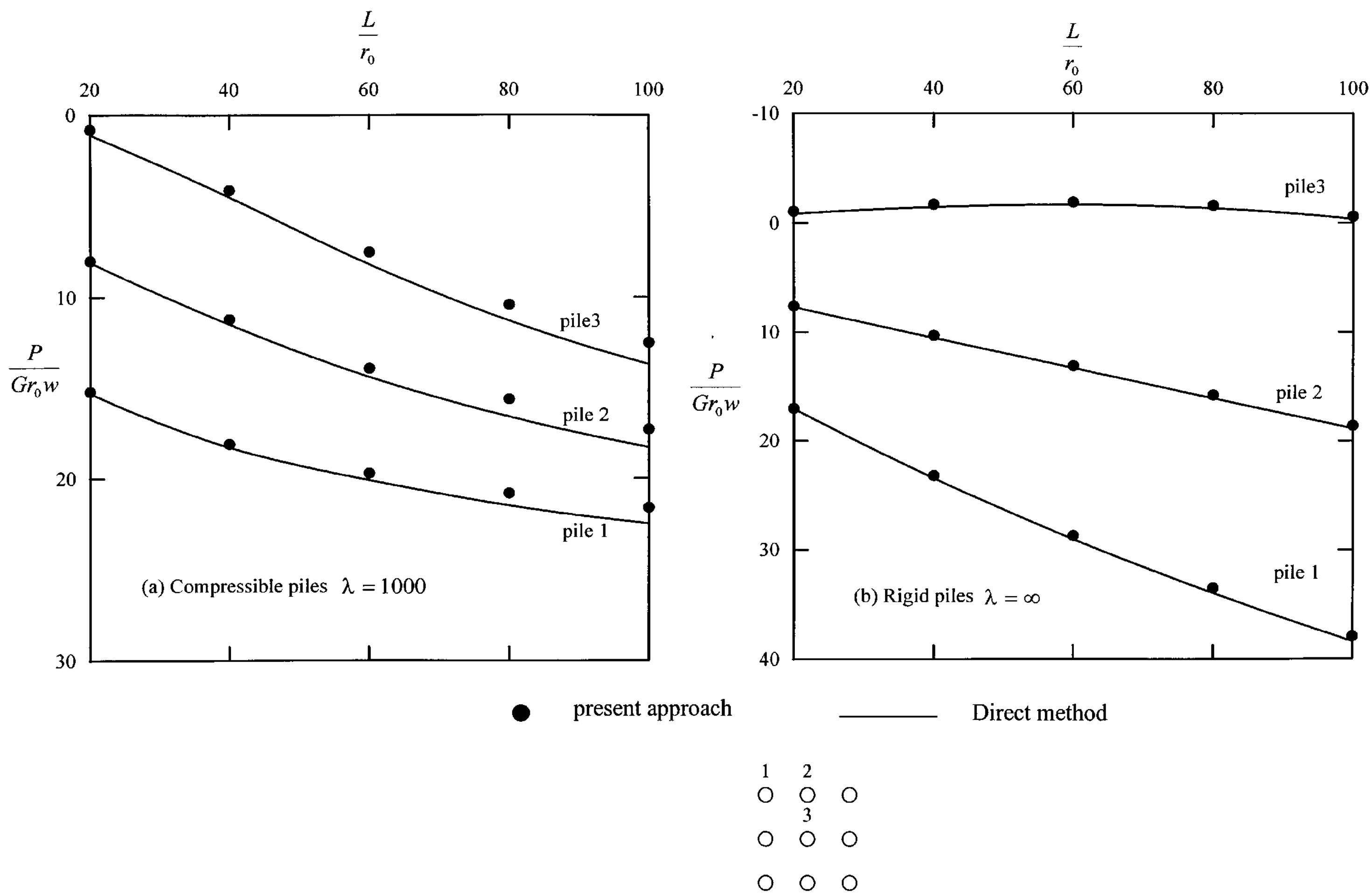

Figure 2. Comparison of distribution of loads to individual piles in 3×3 pile group ($s/r_0 = 5$, $\nu = 0{\cdot}5$, $\beta = 0{\cdot}1$)

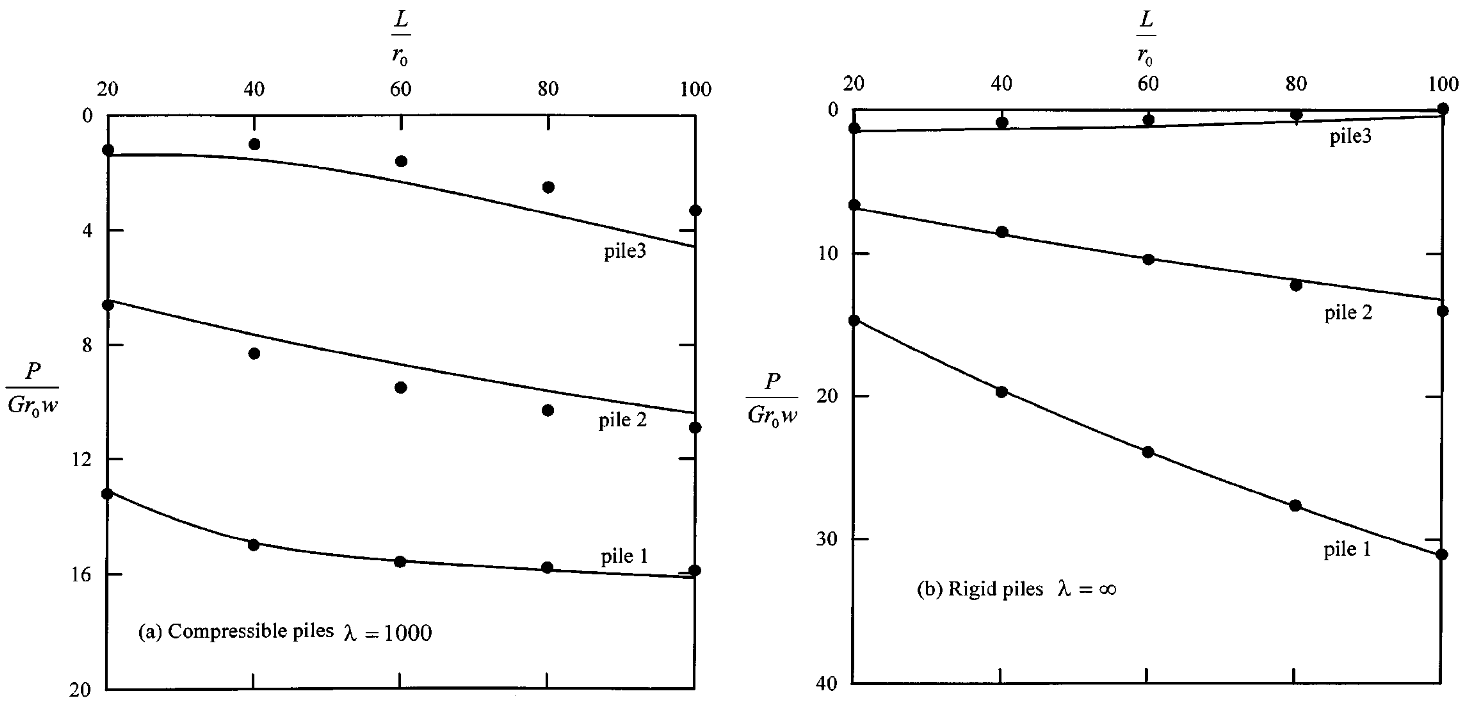

Figure 3. Comparison of distribution of loads to individual piles in 5×5 pile group ($s/r_0 = 5$, $\nu = 0{\cdot}5$, $\beta = 0{\cdot}1$). ● present approach, (——) direct method

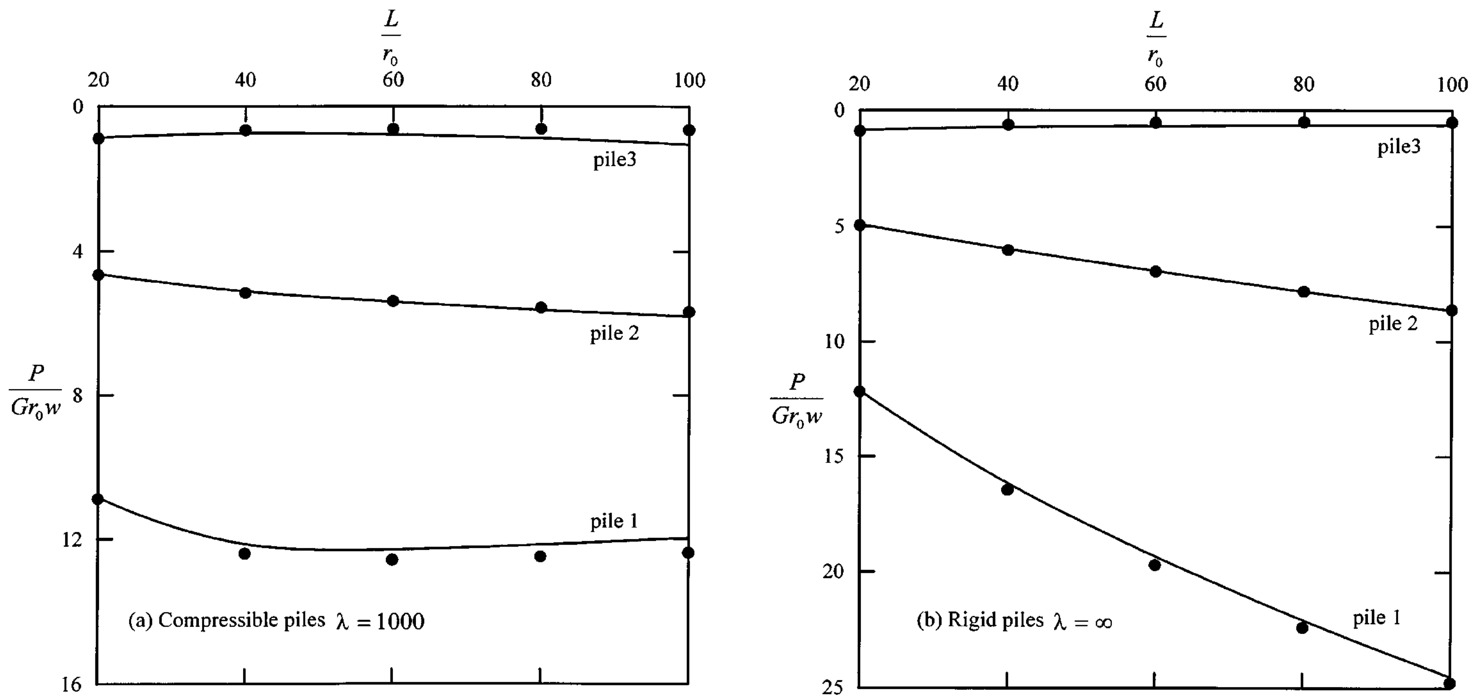

Figure 4. Comparison of distribution of loads to individual piles in 10×10 pile group ($s/r_0 = 5$, $\nu = 0{\cdot}5$, $\beta = 0{\cdot}1$) ● present approach, (——) direct method

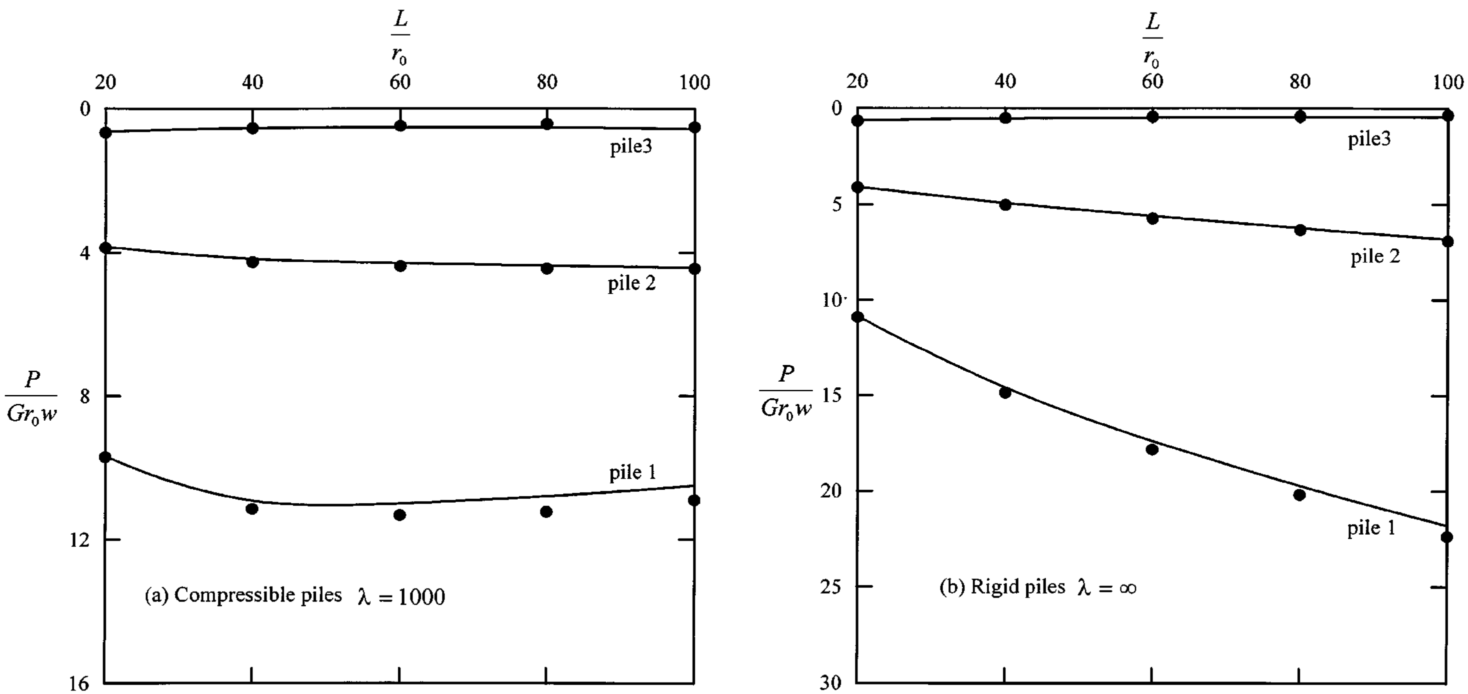

Figure 5. Comparison of distribution of loads to individual piles in 15×15 pile group ($s/r_0 = 5$, $\nu = 0{\cdot}5$, $\beta = 0{\cdot}1$). ● present approach, (——) direct method

Table VII. Normalized stiffness p_{group}/Gr_0w of pile group ($\lambda = 1000$, $L/r_0 = 60$, $s/r_0 = 5$, $v = 0{\cdot}5$ and $\beta = 0{\cdot}1$)

Number of piles	Present approach	Direct method	$1 - \dfrac{\text{Present}}{\text{Direct}}$ (%)
100	320	333	4·0
150	374	388	3·6
200	418	434	3·5
250	456	472	3·3
300	491	507	3·2
350	524	541	3·0

Table VIII. Solution time of the present approach and Direct method ($\lambda = 1000$, $L/r_0 = 60$, $s/r_0 = 5$, $v = 0{\cdot}5$ and $\beta = 0{\cdot}1$)

Pile number	Server†			Personal computer†
	Present approach (min)	Direct Method (min)	$\dfrac{\text{Direct}}{\text{Present}}$	Present approach (min)
100	0·33	26	79	3·5
150	1·25	91	73	12
200	2·97	225	76	28
250	5·73	467	82	54
300	9·95	729	73	94
350	16·0	1605	100	149

† Server type: IBM RISC system/6000

‡ Configuration of the personal computer: (a) 100 MHz Pentium-S, (b) 32 Mb core memory

method and the present approach for the analysis of pile groups in Table VIII. A server IBM RISC system/6000 is used for the comparison since the Direct method requires long solution times for the large pile groups. The solution times of the present approach using a personal computer (PC) are also shown in Table VIII. The CPU type of the PC is 100 MHz Pentium-S and the computer core memory is 32 Mb. Tables VII and VIII show that the present approach can save a lot of solution times, with accuracy within 4 per cent of the solutions of the Direct method. Comparing the solution times of the present approach using the server and the PC in Table VIII, it can be seen that the solution speed of the server is roughly 10 times faster than the PC. Hence, using the Direct method, it would take 10 times more solution time in the PC than the solution times shown in Table VIII using the server. For example, it would take about 1605×10 min, i.e. 11 days, for the PC to solve the problem of 350 piles using the Direct method. In order to obtain the solutions faster, a server is needed in the analysis. As a comparison, it takes only 149 min, i.e. about 2·5 h for the PC to solve the same problem of 350 piles using the present approach while the difference between the two solutions is only 3%. Table VIII also shows that the more piles in a pile group, the more solution time can be saved using the present approach compared to the

Direct method. For instance, using the server, the present approach can save nearly 1600 min for the problem of 350 piles whereas it can save about 220 min for the problem of 200 piles.

The present approach can save not only a significant amount of solution times, but also computer core memory. Using 10 discrete elements (i.e. 11 nodal points) for the individual piles, the computer core memory used for the elements of the stiffness matrix of the pile group with 350 piles in the Direct method is 3850×3850. For larger groups or using more discrete elements for individual piles, this requirement is even more significant. However, for a pile group with piles of same properties, the storage requirement for the elements of the stiffness matrix for each iteration in the present approach is only 11×2 (taking into account the banded nature of the stiffness matrix), irrespective of the size of the group. In the process of iterations, the f_{ij} in Equations (4), (9) and (12) may be used many times. If flexibility coefficients are calculated respectively whenever needed instead of storing them in a computer core memory, solution times spending on calculating the coefficients could be large for a large pile group. For example, using the present method, the solution time for a 100-pile group with $L/r_0 = 60$, $\lambda = 1000$, and $s/r_0 = 5$ is 29 min without storing flexibility coefficients in a computer core memory whereas by storing flexibility coefficients, the solution time is only 3·5 min. Hence, the coefficients f_{ij} should be stored in a computer core memory instead of recalculating them in each iteration.

It should be noted that the pile–soil interaction force at one node of a pile cannot induce additional soil displacements at the other nodes of the same pile. The coefficients f_{ij} is 0 (nodes i, j are associated with the same pile) and need not be stored in the computer memory. Therefore, taking into account the symmetry of the flexibility coefficients f_{ij} and f_{ji}, i.e. $f_{ij} = f_{ji}$ (where i, $j = 1, 2, \ldots, nn$, i, j are not associated with the same pile and nn is the total number of nodes in a pile group), the number of the flexibility coefficients nsize stored in a computer memory is given by

$$\mathrm{nsize} = \frac{\mathrm{nnode}^2 \times n \times (n-1)}{2} \tag{18}$$

where nnode is the number of nodes of each pile, n is the number of piles in a pile group.

Hence, the computer core memory used for the flexibility coefficients is $\frac{1}{2}(3850 \times 3839)$ for the group of 350 piles. It should be noted that the two matrices with the size of 3850×3850 are required using the Direct method. Hence, the present approach just uses less than a quarter of the core memory required in the direct method. For example, using a PC with a core memory of 32 Mb, the present approach can solve problems of more than 350 piles whereas the Direct method can only solve problems of less than 150 piles assuming 10 discrete elements are used for individual piles in the two methods.

COMPARISON WITH FIELD MEASUREMENT

Hooper[14] reported the field measurement of the settlement of the pile raft foundation of Dashwood House, which is situated close to Liverpool Street station in London. Dashwood House is a 61-m high building with 15 floors and a single storey basement. The raft is supported by 462 bored piles which are 15 m long and 0·485 m in diameter. The 21×22 piles are spaced at 1·5-m centres on a square grid. The full details of the field study are given by Green and Hight.[15]

The soil succession is 8 m of fill, sand and gravel, 29 m of London clay, followed by at least 10 m of Woolwich and Reading beds. The base of the raft is founded on gravel which is about 1 m

above the upper clay surface. For simplicity, the raft is considered to sit on the London clay directly in the analysis. The undrained shear strength of the London clay *in situ* can be approximated by the straight line relationship

$$C_u = 150 + 6{\cdot}67 z \,\mathrm{kN/m^2} \tag{19}$$

where z is the depth in meter below foundation level. Adopting a ratio for G/C_u of 200 (Simpson *et al.*) for the London clay, the shear modulus profile is given by

$$G = 30 + 1{\cdot}33 z \,\mathrm{MN/m^2} \tag{20}$$

Since the comparison is for the settlement at the end of construction, a soil Poisson's ratio of 0·5 is assumed. The gross building load is 274 MN, including that of the raft. In the analysis, the interaction transfer parameter β is 0·1. The computed overall stiffness of the pile group is 6872 kN/mm and the computed raft settlement is 40 mm. This compares favourably with the field measurement of 33 mm. Since the piled raft is assumed in the analysis not to be in contact with the ground, the foundation stiffness contributed by the raft is not considered. As was showed by Randolph and Clancy,[17] the overall foundation stiffness for a piled raft is generally not much greater than that of the pile group which supports the raft.

CONCLUSIONS

An iterative approach has been described for the analysis of vertically loaded pile groups. This approach is particularly suited for the analysis of large pile groups. The present approach not only require less computer core memory for computation but also save a lot of solution times compared with the conventional direct methods. The present approach is particularly efficient for the problems of pile groups with piles of different properties and non-uniform arrangements. Comparison of the approach with the Direct method has shown good agreement. A case study also shows that the present approach can reasonably predict the load-settlement behaviour of vertically loaded large pile groups. The present approach can also be extended to deal with pile groups under other loading conditions.

APPENDIX I

Extension of the present approach to take advantage of symmetry of pile groups

In some engineering practices, pile groups are arranged symmetrically. The present approach can be extended to take into account the symmetry of pile groups in order to save solution time.

The plan of a symmetric pile group can be represented schematically in Figure 6. X- and Y-axis are set as two symmetric axes. The piles in the group can be classified into four types of subgroups, namely, Type-1, Type-2, Type-3 and Type-4 subgroup. There are four piles in each individual Type-1 subgroup. In Figure 6, Piles 1–4 form a Type-1 subgroup, say Subgroup 1. Piles of subgroups Type-2 and Type-3 lie on Y- and X-axis respectively. In each Type-2 or Type-3 subgroup, there are two piles, which are symmetric about X- and Y-axis. In Figure 6, Piles 5 and 6 form a Type-2 subgroup, say Subgroup 2 and Piles 7 and 8 form a Type-3 subgroup, say Subgroup 3. There is only one pile in Type-4 subgroup which is at the centre of the pile group. In Figure 6, Pile 9 forms a Type-4 subgroup, say Subgroup 4.

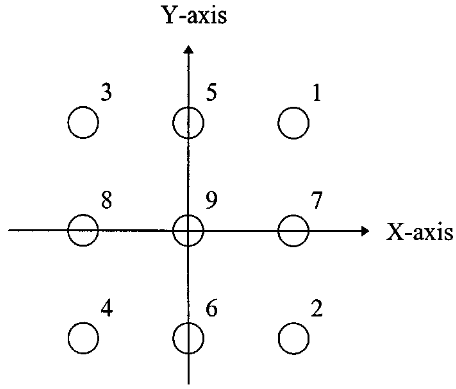

Figure 6. Arrangement of a symmetric pile group

For the moment, there is no rigid pile cap to be used in the group. The consideration of the effect of a rigid pile cap will be given later.

It can be seen that when unit loads act on the heads of all the piles of a subgroup, the settlements of all the piles of any individual subgroup should be the same and can be represented by a pile in the subgroup. A subgroup head flexibility matrix $[FG_{\text{head}}]$ is introduced. The coefficient FG_{ij} in the matrix denotes the head settlement of any pile in subgroup i due to unit vertical loads acting on the head of all piles in subgroup j simultaneously. For a pile group with ns subgroups, the load–deformation relationship can be established as

$$\begin{bmatrix} FG_{11} & FG_{12} & \cdots & FG_{1\text{ns}} \\ FG_{21} & FG_{22} & \cdots & FG_{2\text{s}} \\ \cdots & \cdots & \cdots & \cdots \\ FG_{\text{ns}1} & FG_{\text{ns}2} & \cdots & FG_{\text{nsns}} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ \cdots \\ p_{\text{ns}} \end{Bmatrix} = \begin{Bmatrix} w_1 \\ w_2 \\ \cdots \\ w_{\text{ns}} \end{Bmatrix} \tag{21}$$

or

$$[FG_{\text{head}}]\,\{p_{\text{head}}\} = \{w_{\text{head}}\} \tag{22}$$

where p_j is the vertical force on the head of each pile in subgroup j, w_i is the head settlement of each pile in subgroup i, $i, j = 1, 2, \ldots, ns$, $\{p_{\text{head}}\}$ is the vector of pile head loading on subgroup, $\{w_{\text{head}}\}$ is the vector of pile head settlement of subgroup, and ns is the number of subgroups in the pile group.

For a pile group with ns subgroups, ns calculation steps are needed to obtain subgroup head flexibility coefficients. In each step, unit vertical loads act on all the pile heads of a subgroup while the other piles in the group are free of external loading. For example, when unit loads acts on the heads of Piles 1–4 shown in Figure 6 which belong to Subgroup 1, the settlements of Piles 1–4 are same and can be represented by the settlement of Pile 1, i.e. the subgroup head flexibility coefficient FG_{11} can be represented by the settlement of Pile 1 which is given by superposition from

$$FG_{11} = F_{11} + F_{12} + F_{13} + F_{14} \tag{23}$$

where F_{11}, F_{12}, F_{13} and F_{14} are pile head flexibility coefficients in the matrix $[F_{\text{head}}]$ in Equation (2), which can be determined using the preceding method of analysis in the paper. It may be noted

Table IX. Solution time† of 10×10 pile groups using a personal computer‡ ($\lambda = 1000$, $s/r_0 = 5$, $\nu = 0{\cdot}5$ and $\beta = 0{\cdot}1$)

L/r_0	Solution time (min)	
	Not considering symmetry	Considering symmetry
20	3·5	0·8
40	3·5	0·8
60	3·5	0·8
80	5	1·2
100	5	1·2

† The solution times using the Direct method considering symmetry of pile groups in the personal computer for all the cases are 0·8 min.
‡ Configuration of the personal computer: (a) 100 MHz Pentium-S, (b) 32 Mb core memory

that the physical meanings of FG_{11} and F_{11} are different. Taking into account the symmetry of the response of piles, i.e. $F_{12} = F_{21}$, $F_{13} = F_{31}$ and $F_{14} = F_{41}$, Equation (23) can be rewritten as

$$FG_{11} = F_{11} + F_{21} + F_{31} + F_{41} \tag{24}$$

Similarly, the response of Subgroup 2 with Piles 5 and 6 can be given by

$$FG_{21} = 2 \times (F_{51} + F_{61}) \tag{25}$$

the response of Subgroup 3 with Piles 7 and 8 can be given by

$$FG_{31} = 2 \times (F_{71} + F_{81}) \tag{26}$$

and the response of Subgroup 4 with Pile 9 can be given by

$$FG_{41} = 4 \times F_{91} \tag{27}$$

It can be seen that the influence of unit loads on all the piles of Subgroup 1 can be represented by the pile head flexibility coefficients due to a unit load on Pile 1 only. The other coefficients in matrix $[FG_{\text{head}}]$ can likewise be obtained.

From Equation (22), the subgroup head flexibility matrix $[FG_{\text{head}}]$ can be inverted to give the relationship of the external loads and the pile head settlements as follows:

$$\{p_{\text{head}}\} = [FG_{\text{head}}]^{-1}\{w_{\text{head}}\} \tag{28}$$

or

$$\{p_{\text{head}}\} = [KG_{\text{head}}]\{w_{\text{head}}\} \tag{29}$$

where $[KG_{\text{head}}] = [FG_{\text{head}}]^{-1}$, is the subgroup head stiffness matrix.

Using Equation (29), a prescribed displacement analysis can be implemented to consider the effect of a rigid pile cap.

Table IX compares solution times with and without considering the symmetry of pile groups for the analysis of 10×10 pile groups. Since a pile group with a square layout is not always employed in engineering practices, the 10×10 pile groups are considered as those with rectangular layouts. Hence, a quarter of piles in the groups, i.e. 25 piles, are analysed considering the

symmetry of the pile groups as a typical example. The present approach considering symmetry takes about a quarter of solution times which are spent in the analysis without considering symmetry. Since the two methods give the same normalized stiffness as for all the pile groups, those would not be be given in Table IX. As a comparison, the solution time of the direct method considering symmetry of pile groups is also given.

It may be noted that a pile group with exact symmetric arrangement is not always present in practice in terms of plan layouts of piles, pile properties such as pile diameter and pile penetration depth, and the properties of soils in which piles are embedded. Therefore, an engineering judgement is required while making use of the symmetry of pile groups.

REFERENCES

1. M. Ottaviani, 'Three dimensional finite element analysis of vertically loaded pile groups', *Geotechnique*, **25**(2), 159–174 (1975).
2. R. Butterfield and P. K. Banerjee, 'The elastic analysis of compressible piles and pile groups', *Geotechnique*, **21**(3), 43–60 (1971).
3. H. G. Poulos, 'Analysis of the settlement of pile groups', *Geotechnique*, **18**(4), 449–471 (1968).
4. R. D. Mindlin, 'Force at a point in the interior of a semi-infinite solid', *Physics*, **7**, 195–202 (1936).
5. H. G. Poulos, 'Group factors for pile-deflection estimation', *J. Geotech. Engng. Div., Proc. ASCE*, **105**(GT12), 1489–1509 (1979).
6. P. K. Banerjee and T. G. Davies, 'Analysis of some reported case histories of laterally loaded pile groups', *Proc. Num. Meths in Offshore Piling*, ICE, London, 1980, pp. 83–90.
7. Y. K. Chow, 'Axial and lateral response of pile groups embedded in nonhomogeneous soils', *Int. J. Numer. Analyt. Mech. Geomech.* **11,** 621–638 (1987).
8. M. W. O'Neill, O. I. Ghazzaly and H. B. Ha, 'Analysis of three-dimensional pile groups with nonlinear soil response and pile–soil–pile interaction', Proc. 9th Offshore Technology Conf., Vol. 2, 1977, pp. 245–256.
9. Y. K. Chow, 'Analysis of vertically loaded pile groups', *Int. J. Numer. Analyt. Mech. Geomech.*, **10**(1), 59–72 (1986).
10. M. F. Randolph and C. P. Wroth, 'Analysis of the deformation of vertically loaded piles', *J. Geotech. Engng. Div. Proc. ASCE*, **104**(GT12), 1465–1488 (1978).
11. L. M. Kraft, R. P. Ray and T. Kagawa, 'Theoretical *t–z* curves', *J. Geotech. Engng. Div. Proc. ASCE*, **107**(GT 11), 1543–1561 (1981).
12. Y. K. Chow, 'Iterative analysis of pile–soil–pile interaction', *Geotechnique*, **37**(3), 321–333 (1987).
13. I. M. Smith, *Programming the Finite Element Method, with Application to Geomechanics*, Wiley, Chichester, 1982.
14. J. A. Hooper, 'Review of behaviour of piled raft foundations'. *CIRIA Report 83,* 1979, pp. 45–54.
15. P. A. Green and D. W. Hight, 'The instrumentation of Dashwood House, London', *CIRIA Technical Note 78*, 1976.
16. B. Simpson, G. Calabresi, H. Sommer and M. Wallays, 'Design parameters for stiff clays', *Proc. 7th Eur. Conf. on soil Mech. and Found. Engng.*, Brighton, Vol. 5, 1979, pp. 91–125.
17. M. F. Randolph and P. Clancy, in Van Impe (ed.), *Deep Foundation on Bored and Auger Piles*, Balkema, Rotterdam, 1993.

